The Lying over and Going up Theorems

If $R \subseteq S$ and S is integral over R, how does $\operatorname{Spec} S$ compare to $\operatorname{Spec} R$?

Theorem: Suppose $R \subseteq S$ and S integral over R. If $P \in \operatorname{Spec} R$, there is some $Q \in \operatorname{Spec} S$ with $R \cap Q = P$. Moreover, Q can be chosen to contain any ideal $Q_0 \subseteq S$ that satisfies $R \cap Q_0 \subseteq P$.

Pf: Let $Q_0 \subseteq S$ s.t. $R \cap Q_0 \subseteq P$. We can replace S and R with S/Q_0 and $R/R \cap Q_0$ and assume $Q_0 = 0$. We thus only need to prove the first statement in the theorem.

Let $U = R - P$. Since there is a 1-to-1 correspondence between primes in $U^{-1}S$ and primes in S not meeting U, we can replace S by $U^{-1}S$ and R by $U^{-1}R = R_P$, and assume R is local w/ max'l ideal P.

Any maximal ideal of S containing PS has preimage in R containing P and thus equal to P. Thus, we just need to show PS is contained in a max'l ideal, i.e. $PS \neq S$.

If $PS = S$, then

$$1 = s_1 p_1 + \cdots + s_n p_n$$

where $s_i \in S$, $p_i \in P$. Set $S' = R[s_1, \ldots, s_n] \subseteq S$. Then $1 \in PS'$ so $PS' = S'$. Since S' is integral and finitely generated $/R$, it's finite $/R$.

By Nakayama, $S' = 0$, a contradiction. $\square$

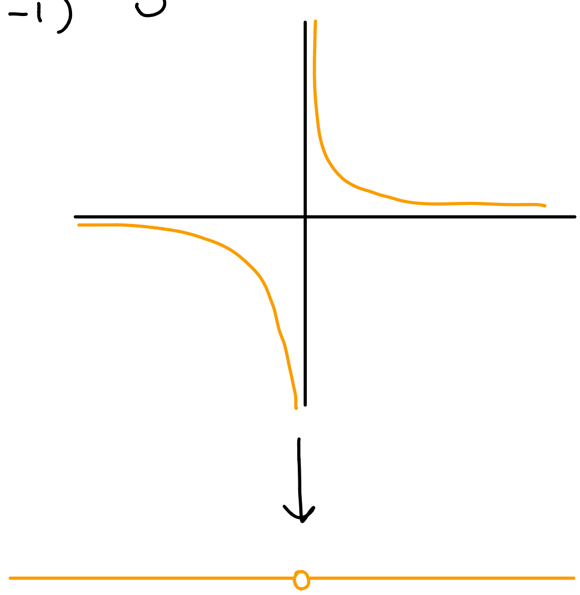
This immediately implies the classical "Lying over" Theorem:

Cor (Lying over): If $R \hookrightarrow S$ is an integral extension, then $\operatorname{Spec} S \to \operatorname{Spec} R$ is surjective.

Note that the integrality hypothesis is necessary:

Ex: Define

$$\varphi : k[t] \hookrightarrow k[x,y]/(xy-1) = S$$
$$t \longmapsto x$$



The nonzero primes in S are of the form $(x-a, y-1/a)$, $a \neq 0$. In particular, the preimages are of the form $(t-a)$, so (t) is not in the image of the Spec map.

Ex: Let R be a ring and $U \subseteq R$ multiplicatively closed. Suppose U contains some nonunit r, and consider $R \to U^{-1}R$.

Let $P \subseteq R$ be a prime ideal containing r. For $Q \in \operatorname{Spec} U^{-1}R$, $\frac{r}{1} \notin Q$, so $Q \cap R \neq P$. Thus

$$\operatorname{Spec} U^{-1}R \to \operatorname{Spec} R$$

is not surjective, so $R \to U^{-1}R$ is only integral if U consists of units (i.e. if it's an isomorphism)

The Theorem also implies the "Going up" theorem:

Cor (Going up): If $R \hookrightarrow S$ is integral and $P_0 \subseteq P_1 \subseteq \cdots \subseteq P_d$ a chain of primes in R, there exists a chain of prime $Q_0 \subseteq Q_1 \subseteq \cdots \subseteq Q_d$ in S such that each Q_i contracts to P_i. i.e. $Q_i \cap R = P_i$.

Pf: First find Q_0 by the Theorem. Then we can find each Q_{i+1} inductively by the theorem since $Q_i \cap R \subseteq P_{i+1}$. $\square$

We will prove the "going down" theorem (which requires more hypotheses) later.

If R and S are integral domains, we can consider the field extension of their fields of fractions, $K(R)$ and $K(S)$:

Lemma: Let $R \subseteq S$ be integral domains and suppose $K(R) \subseteq K(S)$ is an algebraic extension. Then for any nonzero ideal $I \subseteq S$, $I \cap R \neq 0$.

Pf: It suffices to assume $I = (b) \subseteq S$. Then

$$a_n b^n + \cdots + a_1 b + a_0 = 0 \quad \text{w/ } a_i \in K(R),\ a_0 \neq 0.$$

We can multiply through by a common denominator and assume all $a_i \in R$. Thus,

$$\underset{\neq 0}{a_0} \in (b) \cap R. \quad \square$$

Note that this means that if R is a field, any nonzero ideal of S contains <u>all</u> of R, so it contains 1, so S must be a field as well. This gives us the following:

Cor: If $R \subseteq S$ is an integral extension of integral domains, then S is a field $\iff$ R is a field.

Pf: If R is a field, then S is contained in the algebraic closure of R in $K(S)$, so all of $K(S)$ is. Thus, we can apply the lemma, so S is a field.

If S is a field, let $m \subset R$ a max'l ideal. Then there's a prime $Q \subseteq S$ s.t. $Q \cap R = m$. But $Q = 0$, so $m = 0$. Thus R is a field. $\square$

Note that if $P \subseteq S$ is a prime ideal

$$R/P\cap R \subseteq S/P$$

is also an integral extension, so the corollary implies:

Cor: If $R \subseteq S$ an integral extension of integral domains, and $P \in \operatorname{Spec} S$, then $P \subseteq S$ is maximal $\Leftrightarrow$ $P \cap R \subseteq R$ is maximal.

We can also use the lemma to deduce that two points in the fiber of the map on Spec must be incomparable. That is:

Cor: If $R \subseteq S$ is an integral extension, then if

$$Q \subseteq Q_1 \subseteq S$$

are prime ideals s.t. $Q \cap R = Q_1 \cap R$, then $Q = Q_1$.

Pf: Let $P = Q \cap R$. Then $R/P \subseteq S/Q$ is integral.

Thus S/Q is integral over $K(R/P)$, so $K(S/Q)$ is as well (since $K(R/P)$ is a field).

$Q_1/Q \cap R/P = 0$, so the lemma says $Q_1/Q = 0$, so $Q_1 = Q$. $\square$

This allows us to compare the dimensions of Spec R and Spec S:

Def: The (Krull) dimension of R is the supremum of lengths of chains of prime ideals of R. i.e.

$$\dim R := \sup\{d \mid \exists P_i \in \operatorname{Spec} R \text{ s.t. } P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_d\}$$

Ex: 1.) If k is a field, $\dim k = 0$.

2.) In $k[x]$, every nonzero ideal is max'l, so $\dim k[x] = 1$.

3.) In $k[x_1, \dots, x_n]$, $0 \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \dots \subsetneq (x_1, \dots, x_n)$, so $\dim k[x_1, \dots, x_n] \geq n$. We will see later that this is an equality (which is surprisingly hard to show).

Cor: If $R \hookrightarrow S$ is integral, then $\dim R = \dim S$.

Pf: If $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ is a chain of primes in R, then Going up says we can find a chain $Q_0 \subsetneq \cdots \subsetneq Q_n$ in S, so $\dim S \geq \dim R$.

If $Q_0 \subsetneq \cdots \subsetneq Q_n$ is a chain of primes in S, then the previous corollary says $Q_i \cap R \subsetneq Q_{i+1} \cap R$ for each i, so $\dim R \geq \dim S$. $\square$

We will come back to dimension soon after first discussing some important invariants of modules.